Worksheet 2 Solutions

Velocity

The displacement of an object on a line is given by the equation

$$s = 1 + 2t + \frac{1}{4}t^2,$$

where t is in seconds.

1. Find the average velocity in m/sec over each of the following time periods. For parts a, b, and c, give the answer to 4 digits after the decimal point. In part d, h is a nonzero variable.
(a) between $t = 1$ and $t = 1.5$
(b) between $t = 1$ and $t = 1.1$
(c) between $t = 1$ and $t = 1.01$
(d) between $t = 1$ and $t = 1 + h$, where h is a nonzero variable.

Solution.

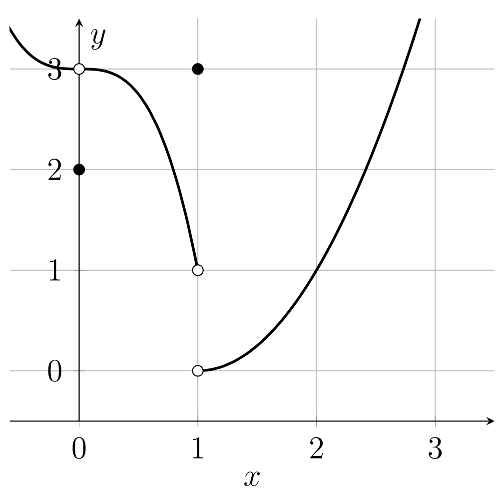
(a) 2.6250 (b) 2.5250 (c) 2.5025 (d) $2.5 + h/4$

2. Find the instantaneous velocity (in m/sec) at time $t = 1$ second.

Solution.
2.5 m/s

The Limit of a Function

3. The graph of $y = f(x)$ is below. Compute each value or explain why it doesn't exist.



(a) $\lim\limits_{x\to 0^-} f(x)$	(b) $\lim\limits_{x\to 0^+} f(x)$	(c) $\lim\limits_{x\to 0} f(x)$	(d) $f(0)$
(e) $\lim\limits_{x\to 1^-} f(x)$	(e) $\lim\limits_{x\to 1^+} f(x)$	(e) $\lim\limits_{x\to 1} f(x)$	(h) $f(1)$
(i) $\lim\limits_{x\to 2^-} f(x)$	(j) $\lim\limits_{x\to 2^+} f(x)$	(k) $\lim\limits_{x\to 2} f(x)$	(l) $f(2)$

Solution.

(a) 3	(b) 3	(c) 3	(d) 2
(e) 1	(f) 0	(g) DNE	(h) 3
(i) 1	(j) 1	(k) 1	(l) 1

The only value that DNE is (g). This is because the left and right limits at 1 do not agree (see parts e and f).

Using Limit Laws

4. Evaluate the limit if it exists using algebra and limit laws. Draw a graph to support your answer.

(a) $\lim\limits_{x\to 1} \dfrac{x+2}{10(x-1)}$ (b) $\lim\limits_{x\to 3} \dfrac{\sqrt{x^2+40}-7}{x-3}$

Solution.

(a) Break up into left and right hand limits:

$$\lim_{x\to 1^+} \frac{x+2}{10(x-1)} = \frac{\lim\limits_{x\to 1^+} x+2}{\lim\limits_{x\to 1^+} 10(x-1)} = \frac{\to 3}{\to \text{small positive } \#} = +\infty$$

$$\lim_{x\to 1^-} \frac{x+2}{10(x-1)} = \frac{\lim\limits_{x\to 1^-} x+2}{\lim\limits_{x\to 1^-} 10(x-1)} = \frac{\to 3}{\to \text{small negative } \#} = -\infty$$

Limits are not equal so $\lim\limits_{x\to 1} \dfrac{x+2}{10(x-1)}$ does not exist.

(b)

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x^2+40}-7}{x-3} &= \lim_{x\to 3} \frac{\sqrt{x^2+40}-7}{x-3}\left(\frac{\sqrt{x^2+40}+7}{\sqrt{x^2+40}+7}\right) \\
&= \lim_{x\to 3} \frac{x^2+40-49}{(x-3)\sqrt{x^2+40}+7} \\
&= \lim_{x\to 3} \frac{x^2-9}{(x-3)\sqrt{x^2+40}+7} \\
&= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)\sqrt{x^2+40}+7} \\
&= \lim_{x\to 3} \frac{(x+3)}{\sqrt{x^2+40}+7} \\
&= \frac{6}{\sqrt{9+40}+7} \\
&= \frac{6}{14} = \frac{3}{7}.
\end{aligned}$$

Continuity

5. Let $f(x) = 1 - \sqrt{1-x^2}$. In class (Section 2.5) we showed that $f(x)$ is continuous in the interval $(-1, 1)$. In other words, we showed that for $-1 < a < 1$

$$\lim_{x \to a} f(x) = f(a).$$

Show that $f(x)$ is continuous at $x = -1$ and $x = 1$ by checking

$$\lim_{x \to -1^+} f(x) = f(-1) \quad \text{(continuity from the right)}$$

and

$$\lim_{x \to 1^-} f(x) = f(1) \quad \text{(continuity from the left).}$$

Solution.

$$f(-1) = 1 - \sqrt{1 - (-1)^2} = 1.$$

$$\begin{aligned}
\lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} 1 - \sqrt{1-x^2} \\
&= 1 - \lim_{x \to -1^+} \sqrt{1-x^2} \\
&= 1 - \sqrt{\lim_{x \to -1^+} (1-x^2)} \\
&= 1 \\
&= f(-1).
\end{aligned}$$

$$f(1) = 1 - \sqrt{1 - (1)^2} = 1.$$

$$\begin{aligned}
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} 1 - \sqrt{1-x^2} \\
&= 1 - \lim_{x \to 1^-} \sqrt{1-x^2} \\
&= 1 - \sqrt{\lim_{x \to 1^-} (1-x^2)} \\
&= 1 \\
&= f(1).
\end{aligned}$$

6. Let

$$g(x) = \begin{cases} x^2 + x & \text{if } x < 1, \\ a & \text{if } x = 1, \\ x - 1 & \text{if } x > 1. \end{cases}$$

(a) Determine the value of a for which g is continuous from the left at 1.
(b) Determine the value of a for which g is continuous from the right at 1.
(c) Is there a value of a for which g is continuous at 1? Explain.

Solution.
(a) Observe that

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^2 + x = 1 + 1 = 2.$$

So for $g(x)$ to be continuous from the left at 1 we need

$$g(1) = \lim_{x\to 1^-} g(x) = 2.$$

(b) Observe that

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} x - 1 = 1 - 1 = 0.$$

So for $g(x)$ to be continuous from the left at 1 we need

$$g(1) = \lim_{x\to 1^+} g(x) = 0.$$

(c) There is no value for a which will make g continuous at 1 since

$$\lim_{x\to 1^-} g(x) \neq \lim_{x\to 1^+} g(x).$$

Limits at Infinity and Horizontal Asymptotes

Find the limit in each case or explain why it does not exist (and if it is $\pm\infty$).

7. $\lim_{x\to\infty} \frac{x^2}{\sqrt{6x^4 - 1}}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{\sqrt{6x^4-1}} &= \lim_{x\to\infty} \frac{x^2}{\sqrt{6x^4-1}} \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) \\
&= \lim_{x\to\infty} \frac{x^2}{\sqrt{6x^4-1}} \left(\frac{\frac{1}{x^2}}{\frac{1}{\sqrt{x^4}}}\right) \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{6x^4-1}{x^4}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{6 - 1/x^4}} \\
&= \frac{1}{\sqrt{6-0}} \\
&= \frac{1}{\sqrt{6}}.
\end{aligned}$$

8. $\lim_{x\to\infty} \dfrac{10000x}{x^3+x}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{10000x}{x^3+x} &= \lim_{x\to\infty} \frac{10000x}{x^3+x}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right) \\
&= \lim_{x\to\infty} \frac{10000/x^2}{1+1/x^2} \\
&= \frac{0}{1+0} \\
&= 0.
\end{aligned}$$

Application of Limits

$*$ 9. In Einstein's theory of relativity [ever see the movie Interstellar?], the Time Dilation Formula is given by

$$T = T_0\sqrt{1 - v^2/c^2}.$$

This formula expresses the elapsed time T of a clock as a function of its velocity v with respect to an observer, where T_0 is the elapsed time of the clock at rest and c is the speed of light.
Find

$$\lim_{v\to c^-} T.$$

Why is a left-hand limit necessary? [In other words, why can't you take the right-hand limit?]

Physical Meaning: This formula represents a consequence of Einstein's Theory of Relativity called Time Dilation. Basically, it says that the as your velocity increases towards the speed of light, less time will go by for you than for a stationary observer. For example, if you were to ride in a rocket ship going VERY fast around the earth, say 99% the speed of light, 1 year for you would be approximately 7 years for the poor people back on earth.

Solution.
Using limit laws,

$$\begin{aligned}
\lim_{v\to c^-} T &= \lim_{v\to c^-} T_0\sqrt{1 - v^2/c^2} \\
&= T_0\sqrt{1 - \lim_{v\to c^-} v^2/c^2} \\
&= T_0\sqrt{1-1} \\
&= 0.
\end{aligned}$$

It is necessary to take a left-hand limit for exactly the same reason it is necessary to take the left hand limit in #5 above. If you approach c from the right then $\lim_{v\to c^+} v^2/c^2 > 1$. This would mean $1 - \lim_{v\to c^+} v^2/c^2 < 0$ and you cannot take the square root of a negative number. In other words, the function $T(v) = T_0\sqrt{1 - v^2/c^2}$ is not defined for $v > c$.