Worksheet 3 Solutions

Using the Limit Definition of The Derivative

1. Write down the limit definition of the derivative. What are two meanings of the derivative that we stated in class?

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

The following are all meanings of the derivative that were mentioned in class:

1. Slope of the tangent line
2. Limit of the slopes of secant lines
3. Instantaneous rate of change of y with respect to x
4. If $f(x)$ represents the position of an object then $f'(x)$ is the object's velocity

2.Find the derivative of the following functions using the limit definition of the derivative.

(a) $f(t) = 2.5t^2 + 6t$

(b) $f(t) = \dfrac{1}{\sqrt{t}}$

Solution.
(a)

$$\begin{aligned}
f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h \to 0} \frac{[2.5(t+h)^2 + 6(t+h)] - [2.5t^2 + 6t]}{h} \\
&= \lim_{h \to 0} \frac{2.5t^2 + 5ht + 2.5h^2 + 6t + 6h - 2.5t^2 - 6t}{h} \\
&= \lim_{h \to 0} \frac{5ht + 2.5h^2 + 6h}{h} \\
&= \lim_{h \to 0} 5t + 2.5h + 6 \\
&= 5t + 6.
\end{aligned}$$

(b)

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\sqrt{t}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t+h}\sqrt{t}} \\
&= \lim_{h\to 0} \frac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t+h}\sqrt{t}} \left(\frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}\right) \\
&= \lim_{h\to 0} \frac{t-(t+h)}{h\sqrt{t+h}\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{t+h}\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{t+h}\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} \\
&= \frac{1}{\sqrt{t+0}-\sqrt{t}\left(\sqrt{t}+\sqrt{t+0}\right))} \\
&= -\frac{1}{t\left(2\sqrt{t}\right)} \\
&= -\frac{1}{2t^{3/2}}.
\end{aligned}$$

Computing Derivatives using the Power, Product, and Quotient Rules

Compute the derivative of the functions below using rules up through Section 3.2 (power rule, sum rule, product rule, quotient rule).

Remark. Although the derivatives of $1/x = x^{-1}$ and $\sqrt{x} = x^{1/2}$ are special cases of the power rule, they come up often enough that they are worth memorizing so that you know them immediately when needed:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \quad \text{and} \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

3. $f(x) = 7x^3 - 5x + 8$

Solution.

$$f'(x) = 21x^2 - 5.$$

4. $f(x) = \sqrt{x} \cdot x^4$

Solution.

First,

$$f(x) = \sqrt{x} \cdot x^4 = x^{4+1/2} = x^{9/2}.$$

Therefore, by the power rule

$$f'(x) = \frac{9}{2}x^{9/2-1} = \frac{9}{2}x^{7/2}.$$

5. $f(x) = \dfrac{1}{x} + \dfrac{1}{1-x}$

Solution.
Take the derivative of each term individually:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{0(1-x) - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Hence,

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2}.$$

6. $f(x) = \dfrac{ax}{x+b}$, a and b are constants

Solution.
By the Quotient Rule,

$$f'(x) = \frac{(a)(x+b) - (ax)(1)}{(x+b)^2} = \frac{ax + ab - ax}{(x+b)^2} = \frac{ab}{(x+b)^2}.$$

7. $f(x) = \dfrac{e^x}{1+e^x}$

Solution.
By The Quotient Rule,

$$f'(x) = \frac{e^x(1+e^x) - e^x(e^x)}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

8. $f(x) = \dfrac{e^x}{x^n}$, n a constant.

Solution.
Method 1 (Product Rule):
Write $f(x) = e^x x^{-n}$ and then use the product rule

$$
\begin{aligned}
f'(x) &= e^x(x^{-n}) + e^x(-nx^{-n-1}) \\
&= e^x x^{-n} - ne^x x^{-n-1} \\
&= e^x x^{-n}(1 - nx^{-1}) \\
&= \frac{e^x(1 - nx^{-1})}{x^n}
\end{aligned}
$$

Method 2 (Quotient Rule):
By the Quotient Rule,

$$
\begin{aligned}
f'(x) &= \frac{e^x(x^n) - e^x(nx^{n-1})}{(x^n)^2} \\
&= \frac{e^x x^n - ne^x x^{n-1}}{x^{2n}} \\
&= \frac{e^x x^n(1 - nx^{-1})}{x^{2n}} \\
&= \frac{e^x(1 - nx^{-1})}{x^n}
\end{aligned}
$$

9. $f(x) = e^x x^n$, n a constant

Solution.
By the Product Rule,

$$
\begin{aligned}
f'(x) &= e^x(x^n) + e^x(nx^{n-1}) \\
&= e^x x^n + ne^x x^{n-1} \\
&= e^x x^n(1 + nx^{-1}).
\end{aligned}
$$