Worksheet 4: More Derivatives (3.3-3.6) Solutions

Name: ________________________ Due: June 15th by 3PM via HuskyCT

Compute the derivative of the functions below using **any** of the differentiation rules up through Section 3.6.

1. $f(x) = \dfrac{\sin(x)}{1+\sin(x)}$

Solution.

$$f'(x) = \frac{(1+\sin x)\cos x - (\sin x)(\cos x)}{(1+\sin x)^2} = \frac{\cos x}{(1+\sin x)^2}$$

2. $f(x) = \sin(x)\cos(x)$

Solution.

$$f'(x) = (\sin x)(-\sin x) + (\cos x)(\cos x) = -\sin^2 x + \cos^2 x$$

3. $f(x) = x^n \cos(x)$, n a constant.

Solution.

$$f'(x) = nx^{n-1}\cos x + x^n(-\sin x) = x^{n-1}(n\cos x - x\sin x)$$

4. $f(x) = \dfrac{\tan(x)}{x^2+1}$

Solution.

$$f'(x) = \frac{(\sec^2 x)(x^2+1) - (\tan x)(2x)}{(x^2+1)^2}$$

5. $y = (x^3 - x + 1)^{10}$

Solution.

$$f'(x) = 10(x^3 - x + 1)^9(3x^2 - 1)$$

6. $y = \sqrt{x^3 + 4x}$

Solution.

$$f'(x) = \frac{1}{2}(x^3 + 4x)^{-1/2}(3x^2 + 4) = \frac{3x^2 + 4}{2\sqrt{x^3 + 4x}}$$

7. $y = 3^{x^4} \cos(x)$

Solution.

$$f'(x) = (3^{x^4} \ln(3) \cdot 4x^3)(\cos x) + (3^{x^4})(-\sin x) = 3^{x^4}(4\ln(3)x^3 \cos x - \sin x)$$

8. $f(x) = \ln(\sqrt{x})$

Solution.

$$f'(x) = \frac{1}{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}\sqrt{x}} = \frac{1}{2x}$$

9. $f(x) = x^{3x^2}$

Solution.
Let $y = x^{3x^2}$. Then

$$\ln(y) = \ln(x^{3x^2}) = 3x^2 \ln(x).$$

Differentiating both sides

$$\frac{y'}{y} = (6x)(\ln(x)) + (3x^2)\left(\frac{1}{x}\right) = 6x\ln(x) + 3x.$$

Multiplying both sides by y

$$y' = y(6x\ln(x) + 3x) = x^{3x^2}(6x\ln(x) + 3x) = 3x^{3x^2+1}(2\ln(x) + 1)$$

Use **implicit differentiation** to differentiate y with respect to x. Your formula for y' may involve both x and y.

10. $x^2y - axy^3 = x + y$, where a is a constant

Solution.

$$\begin{aligned}
(2xy + x^2y') - (ay^3 + ax \cdot 3y^2y') &= 1 + y' \\
x^2y' - 3axy^2y' - y' &= 1 - 2xy + ay^3 \\
y'(x^2 - 3axy^2 - 1) &= 1 - 2xy + ay^3 \\
y' &= \frac{1 - 2xy + ay^3}{x^2 - 3axy^2 - 1)}
\end{aligned}$$

11. $e^{xy} = x^2 + y^2$

Solution.

$$\begin{aligned} e^{xy} \cdot \frac{d}{dx}(xy) &= 2x + 2yy' \\ e^{xy}(1 \cdot y + x \cdot y') &= 2x + 2yy' \\ e^{xy}y + e^{xy}xy' &= 2x + 2yy' \\ e^{xy}xy' - 2yy' &= 2x - e^{xy}y \\ y'(e^{xy}x - 2y) &= 2x - e^{xy}y \\ y' &= \frac{2x - e^{xy}y}{e^{xy}x - 2y} \end{aligned}$$

12. $\sin(x + y) = x + \cos(3y)$

Solution.

$$\begin{aligned} \cos(x + y)(1 + y') &= 1 - \sin(3y)(3y') \\ \cos(x + y) + \cos(x + y)y' &= 1 - 3\sin(3y)y' \\ \cos(x + y)y' + 3\sin(3y)y' &= 1 - \cos(x + y) \\ y'(\cos(x + y) + 3\sin(3y)) &= 1 - \cos(x + y) \\ y' &= \frac{1 - \cos(x + y)}{\cos(x + y) + 3\sin(3y)} \end{aligned}$$

13. **Extra Credit:** Show that for any $x > 0$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Hint: Manipulate the fraction and the exponent algebraically and use the fact you learned from class (Section 3.6) that $\lim\limits_{n\to\infty} (1 + \frac{1}{n})^n = e$.

Solution.

Just using algebra we can manipulate the expression $\left(1 + \frac{x}{n}\right)^n$ to be

$$\left(1 + \frac{x}{n}\right)^n = \left(1 + \frac{1}{n/x}\right)^n = \left(\left(1 + \frac{1}{n/x}\right)^{n/x}\right)^x.$$

Notice that nothing in the problem was changed and that this can be simplified right back down to what we started with.

Now let $m = n/x$ so that

$$\left(\left(1 + \frac{1}{n/x}\right)^{n/x}\right)^x = \left(\left(1 + \frac{1}{m}\right)^m\right)^x.$$

Hence, using the fact that $\lim_{m\to\infty}(1+\frac{1}{m}) = e$, we get

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^m\right)^x \\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x \\
&= (e)^x \\
&= e^x.
\end{aligned}$$

Note that in the above computation we were allowed to move the limit inside the parenthesis since $\left((1+\frac{1}{m})^m\right)^x$ is an exponential function [of the form b^x] and exponential functions are continuous. [Recall from earlier in the course you can move limits inside continuous functions.]