Solutions

Rates of Change

1. The height (in meters) of a projectile shot vertically upward from a point 4 m above ground level with an initial velocity of 23.5 m/s is $h = 4 + 23.5t - 4.9t^2$ after t seconds. (Round your answers to two decimal places.)

(a) Find the velocity after 2s.

(b) When does the projectile reach its maximum height? (*Recall*: A projectile reaches max height when its velocity is 0.)

(c) What is the maximum height?

(d) When does the projectile hit the ground?

(e) With what velocity does it hit the ground?

Solution.

(a) $v(t) = h'(t) = 23.5 - 9.8t \implies v(2) = 23.5 - 9.8(2) = 3.9$ m/s

(b) $v(t) = 23.5 - 9.8t = 0 \implies t = 23.5/9.8 = \frac{235}{98} \approx 2.398$ s

(c) $h\left(\frac{235}{98}\right) = 4 + 23.5\left(\frac{235}{98}\right) - 4.9\left(\frac{235}{98}\right)^2 \approx 32.176$ m

(d) $h(t) = 4 + 23.5t - 4.9t^2 = 0 \implies t \approx -0.1646$ and $t \approx 4.9605$. So the projectile hits the ground at $t = 4.9605$ s

(e) $v(4.9605) = 23.5 - 9.8(4.9605) = -25.1129$ m/s

Exponential Growth

2. The element Adamantium has a half-life of 3 years.

(a) Suppose that you have an initial 14 kg mound of Adamantium. Find a formula for the mass of the sample that remains after t years.

(b) Wolverine needs at least 8 kg of Adamantium to make new claws. How many years will he have to make them until the Adamantium shrinks to 8 kg? Round your answer to the nearest tenth.

Solution.

(a) $m(t) = m(0)e^{kt} = 14e^{kt}$. To find k use the information about the half-life:

$$\begin{aligned} m(3) &= \frac{1}{2}(14) \\ 14e^{k(3)} &= 7 \\ e^{3k} &= \frac{1}{2} \\ k &= \frac{1}{3}\ln\left(\frac{1}{2}\right) \\ &= -\frac{\ln(2)}{3} \end{aligned}$$

So the formula for the mass of the sample that remains after t years is

$$m(t) = 14e^{-\frac{\ln(2)}{3}t}$$

(b)

$$m(t) = 14e^{-\frac{\ln(2)}{3}t} = 8$$

Solving the equation for t yields $t = 2.1$ years.

3. Starbucks serves coffee at 180°F and room temperature in Starbucks is 75°F.

(a) If the coffee at time $t = 0$ (in minutes) has temperature 180°, write down a differential equation for its temperature $T(t)$ at time $t > 0$. Your equation will have an unknown constant "k" that depends on the coffee.

(b) Find a formula for the temperature at time t.

(c) If the coffee is 120° after 10 minutes, how much additional time (from this moment) will it take for the temperature of the coffee to reach 100°? Round your answer to the nearest tenth.

Solution.

(a) The differential equation is

$$\frac{dT}{dt} = k(T - 75), \text{ for some } k < 0.$$

(b) The solution of the above differential equation is

$$\begin{aligned} T(t) - T_s &= [T(0) - T_s]e^{kt} \\ T(t) - 75 &= [180 - 75]e^{kt} \\ T(t) &= 105e^{kt} + 75 \end{aligned}$$

To find k we use the given information that the coffee is 120° after 10 minutes:

$$T(10) = 105e^{k(100)} + 75 = 100$$

$$10k = \ln\left(\frac{45}{105}\right)$$

$$k \approx -0.847$$

Hence, the equation for the temperature at time t is

$$T(t) = 105e^{-0.0847t} + 75$$

(c) To find the additional time t when the temperature is 100° we want to solve

$$\begin{aligned} T(t+10) = 105e^{-0.0847(t+10)} + 75 &= 100 \\ -0.0847(t+10) &= \ln\left(\frac{25}{105}\right) \\ t &= 6.94 \approx 6.9 \text{ minutes} \end{aligned}$$

Alternatively: You can solve

$$T(t) = 105e^{-0.0847t} + 75 = 100$$

for t and then **subtract 10 minutes** at the end. Either way, you need to make sure you account for the fact that you are looking for how many *additional* minutes it takes after 10 minutes has already passed.

Related Rates

4. The radius r of a spherical balloon is expanding at the rate of 10 in/min.
(a) Determine the rate at which the volume V changes with respect to time, in in^3/min, at the instant when $r = 4$ inches. Round your answer to the nearest integer. Recall that $V = \frac{4}{3}\pi r^3$.
(b) Determine the rate at which the surface area S changes with respect to time, in in^2/min, where $r = 4$ inches. Round your answer to the nearest integer. Recall that $S = 4\pi r^2$.
(c) If the radius doubles, does dV/dt double? Does dS/dt double?

Solution.
(a) We are given $\dfrac{dr}{dt} = 10$ and we want to find $\dfrac{dV}{dt}$ when $r = 4$.

$$\begin{aligned} V = \frac{4}{3}\pi r^3 \implies \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ \implies \frac{dV}{dt} &= 4\pi(4)^2 10 \\ \implies \frac{dV}{dt} &= 640\pi \text{ in}^3/\text{min} \end{aligned}$$

(b)

$$\begin{aligned} S = 4\pi r^2 \implies \frac{dS}{dt} &= 8\pi r \frac{dr}{dt} \\ \implies \frac{dS}{dt} &= 8\pi(4)(10) \\ \implies \frac{dS}{dt} &= 320\pi \text{ in}^2/\text{min} \end{aligned}$$

(c) If the radius double, then the dV/dt quadruples:

$$\frac{dV_1}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV_2}{dt} = 4\pi(2r)^2\frac{dr}{dt} = 4\pi(4r^2)\frac{dr}{dt} = 4\left(4\pi r^2\frac{dr}{dt}\right) = 4\frac{dV_1}{dt}$$

However, dS/dt does double:

$$\frac{dS_1}{dt} = 8\pi r\frac{dr}{dt}$$

$$\frac{dS_2}{dt} = 8\pi(2r)\frac{dr}{dt} = 2\left(8\pi r\frac{dr}{dt}\right) = 2\frac{dS_1}{dt}$$

5. Water is flowing into an upside-down right circular cone with height 5 m and radius 2 m at the top.

(a) Suppose that the water fills the cone up to a height of h meters with a radius of r meters at its surface. Draw a diagram of this. Find a formula for the volume V of water in terms of only the variable h (no r in the formula). Recall that the volume of a right-circular cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.

(b) Find dV/dt (again, your equation should not have r in it).

(c) If water is being pumped into the tank at a rate of 3 m^3/min, how quickly is the water level is rising when the water is 2 m deep. Round your answer to the nearest tenth.

Solution.

The diagram is exactly the same as the example we did in Section 3.9. Using similar triangles we can setup a ratio:

$$\frac{r}{h} = \frac{2}{5} \implies r = \frac{2}{5}h$$

Substitute this into the volume formula:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{1}{3}\pi\frac{4}{25}h^3 = \frac{4\pi}{75}h^3.$$

(b) Taking the derivative of the equation for V we just found:

$$\frac{dV}{dt} = \frac{12\pi}{75}h^2\frac{dh}{dt}$$

(c) We are given $\frac{dV}{dt} = 3$ and want to find $\frac{dh}{dt}$ when $h = 2$:

$$\frac{dV}{dt} = \frac{12\pi}{75}h^2\frac{dh}{dt}$$

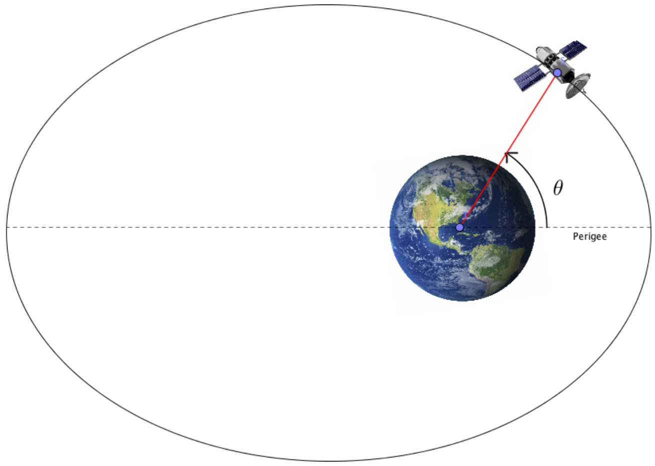
$$3 = \frac{12\pi}{75}(3)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{12\pi} \approx 0.66315 \text{ m/min}$$

6. A satellite is in orbit around earth. The distance from the center of the earth to the satellite is given by the equation

$$r = \frac{4995}{1 + .12\cos(\theta)} \text{ miles},$$

where θ represents the angle the satellite forms with the "Perigee" of the earth.



Find the rate at which the distance is changing at the instant where $\theta = \dfrac{2\pi}{3}$ radians and $\dfrac{d\theta}{dt} = \dfrac{1}{60}$ radians/min.

Solution.
We are given

$$r = \frac{4995}{1 + .12\cos(\theta)} = 4995(1 + .12\cos(\theta))^{-1}.$$

Taking the derivative of both sides with respect to t [using the chain rule] yields

$$\begin{aligned}\frac{dr}{dt} &= -4995(1 + .12\cos(\theta))^{-2} \cdot (-.12\sin(\theta)\frac{d\theta}{dt}) \\ &= \frac{4995 \cdot (.12\sin(\theta))}{(1 + .12\cos(\theta))^2}\frac{d\theta}{dt} \\ &= \frac{599.4\sin(\theta)}{(1 + .12\cos(\theta))^2}\frac{d\theta}{dt}.\end{aligned}$$

So when $\theta = \dfrac{2\pi}{3}$ and $\dfrac{d\theta}{dt} = \dfrac{1}{60}$ we have

$$\begin{aligned}\frac{dr}{dt} &= \frac{599.4\sin\left(\frac{2\pi}{3}\right)}{(1 + .12\cos(\frac{2\pi}{3}))^2}\frac{1}{60} \\ &= \frac{599.4\left(\frac{\sqrt{3}}{2}\right)}{60(1 + .12\left(-\frac{1}{2}\right))^2} \\ &\approx 9.7913 \text{ mi/min.}\end{aligned}$$

Linear Approximations

7. (a) Find the linearization of $f(x) = \sqrt[3]{x+5}$ at 3.

   (b) Use the linear approximation obtained in part a to approximate $\sqrt[3]{7}$.

Solution.

(a) $L(x) = f'(a)(x-a) + f(a) = f'(3)(x-3) + f(3)$.

$$f'(x) = \frac{1}{3}(x+5)^{-2/3} \implies f'(3) = \frac{1}{3}(3+5)^{-2/3} = \frac{1}{12}$$

$$f(3) = \sqrt[3]{3+5} = 2$$

$$L(x) = \frac{1}{12}(x-3) + 2 = \frac{x}{12} + \frac{7}{4}$$

(b) $x = 2 \implies f(2) = \sqrt[3]{2+5} = \sqrt[3]{7}$. So $L(2) = \dfrac{2}{12} + \dfrac{7}{4} = \dfrac{23}{12} \approx 1.91\overline{6}$

Derivatives and Graphs

8. Let $f(x) = x^4 - 8x^2$. Use calculus to find
(a) any vertical and horizontal asymptotes.
(b) the critical numbers of $f(x)$.
(c) the open intervals where $f(x)$ is increasing or decreasing.
(d) the local maximum and minimum values of f.
(e) the intervals of concavity and inflection points.
(f) Sketch a plot of $f(x)$ using the information you found in parts (a)-(d).

Solution.
(a) **Vertical Asymptotes:** None.
Horizontal Asymptotes:

$$\lim_{x\to\infty} x^4 - 8x^2 = \infty \textbf{ and } \lim_{x\to-\infty} x^4 - 8x^2 = \infty.$$

So there are no horizontal asymptotes.

(b)

$$\begin{aligned} f'(x) = 4x^3 - 16x &= 0 \\ 4x(x^2-4) &= 0 \\ x^2 - 4 &= 0 \quad \boxed{x=0} \\ (x-2)(x+2) &= 0 \\ x = 2, x &= -2 \end{aligned}$$

Critical Numbers: $x = -2, 0, 2$.

(c)

Interval	Test #	$f'(x) = 4x^3 - 16x$	$f(x)$ is ...
$x < -2$	-3	(-)	decreasing
$-2 < x < 0$	-1	(+)	increasing
$0 < x < 2$	1	(-)	decreasing
$x > 2$	3	(+)	increasing

Intervals of increase: $(-2, 0)$ and $(2, \infty)$
Intervals of decrease: $(-\infty, -2)$ and $(0, 2)$

(d) Using the table above and the first derivative test we conclude that: $x = -2$ is a local min with the value $f(-2) = -16$, $x = 0$ is a local max with the value $f(0) = 0$, and $x = 2$ is a local min with the value $f(2) = -16$.

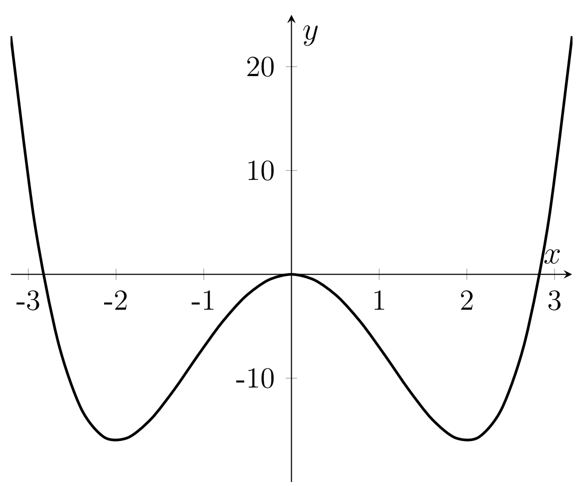
(e) Set second derivative equal to 0 and solve:

$$f''(x) = 12x^2 - 16 = 0 \implies x = -\frac{2}{\sqrt{3}} \approx -1.1547,\ x = \frac{2}{\sqrt{3}} \approx 1.1547$$

Interval	Test #	$f''(x) = 12x^2 - 16$	$f(x)$ is concave ...
$x < -\frac{2}{\sqrt{3}}$	-2	$(+)$	up
$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	0	(-)	down
$x > \frac{2}{\sqrt{3}}$	2	$(+)$	up

Hence, there are two inflection points: At $x = -\frac{2}{\sqrt{3}}$ the graph changes from concave up to concave down and at $x = \frac{2}{\sqrt{3}}$ the graph changes from concave down to concave up.
Inflection Points: $x = -\frac{2}{\sqrt{3}},\ x = \frac{2}{\sqrt{3}}$

(f)



Extra Credit: Your car GPS uses a satellite in space in order to determine where you are and how to get you to your desired location on earth. As seen in the figure below (Figure 1), the GPS device measures the distance h between your car and the satellite. The GPS then uses this information to calculate L, the distance on earth between your car and destination. In other words, the GPS computes L **as a function of** h. However, there is usually some error $\triangle h$ in the measurement of h. For example, the satellite may measure $h = 42,000$ kilometers, but the actual distance between your car and the satellite is $41,975$ kilometers. Therefore, we need to know how accurate the calculation of L is. To determine this we need to find $\dfrac{\triangle L}{\triangle h} \approx \dfrac{dL}{dh}$.

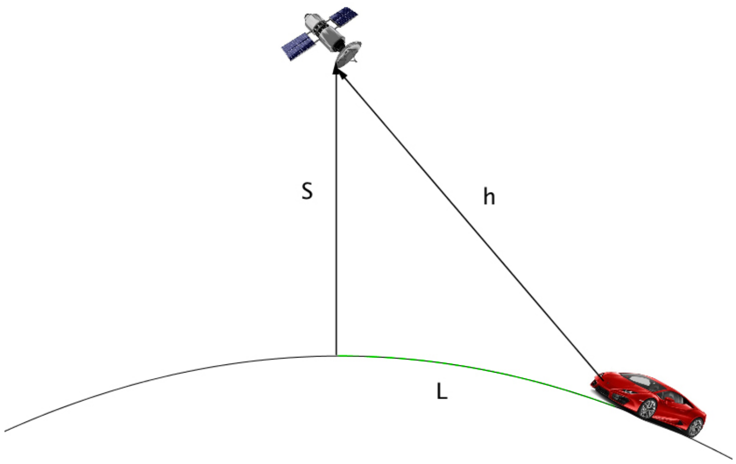


Figure 1

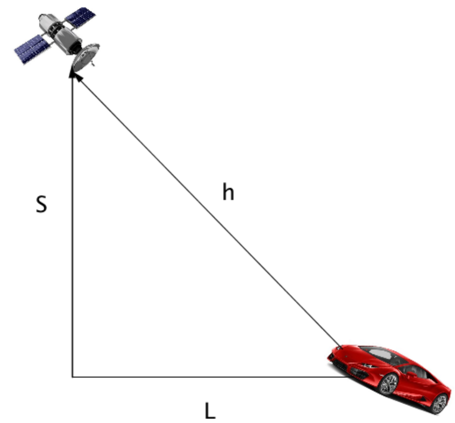


Figure 2

To make things easier, assume the earth is flat (Figure 2). Let $S = 38000$ kilometers (vertical distance between the satellite and earth) and $h = 42,000$ kilometers (the measured distance between your car and the satellite).

(a) Use implicit differentiation with respect to h to calculate $\frac{dL}{dh}$.

(b) Suppose that the actual distance between your car and the satellite is $41,975$ kilometers. Find $\frac{\triangle L}{\triangle h}$ and compare this with $\frac{dL}{dh}$.

Solution.

(a) Observe that $S = 38000$ (vertical distance between the satellite and earth) is **constant**. Hence,

$$S^2 + L^2 = h^2 \implies (38000)^2 + L^2 = h^2.$$

Taking the derivative of both sides with respect to h yields

$$0 + 2L\frac{dL}{dh} = 2h \implies \frac{dL}{dh} = \frac{2h}{2L} = \frac{h}{L}.$$

Hence, when $h = 42000$ we have

$$\frac{dL}{dh} = \frac{42000}{L}$$

We can find L using the pythagorean formula:

$$(38000)^2 + L^2 = (42000)^2 \implies L = 8000\sqrt{5}$$

Thus,

$$\frac{dL}{dh} = \frac{42000}{L} = \frac{42000}{8000\sqrt{5}} = \frac{21}{4\sqrt{5}} \approx 2.3479$$

(b) We are given $h_{calculated} = 42000$ and $h_{actual} = 41975$. Using the pythagorean formula we find that:

$$L_{calculated} = 8000\sqrt{5} \approx 17888.5438 \text{ km and } L_{actual} = 25\sqrt{508641} \approx 17829.7679 \text{ km}$$

$$\frac{\triangle L}{\triangle h} = \frac{L_{actual} - L_{calculated}}{h_{actual} - h_{calculated}} = \frac{17829.7679 - 17888.5438}{41975 - 42000} \approx 2.3510$$