

Worksheet 6: (4.4-5.2) Solutions

Indeterminate Forms and L'Hospital's Rule

1. State each limit's indeterminate form and then compute the limit. If l'Hospital's Rule is needed more than once, try to simplify the expression before applying it.

(a) $\lim_{x \rightarrow 0} \frac{(x+1)^9 - 9x - 1}{x^2}$

Solution.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(x+1)^9 - 9x - 1}{x^2} &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0} \frac{9(x+1)^8 - 9}{2x} \\ &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0} \frac{72(x+1)^7}{2} \\ &= 36.\end{aligned}$$

(b) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{e^{9x} - e^{2x}}$

Solution.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(3x)}{e^{9x} - e^{2x}} &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0} \frac{\cos(3x) \cdot 3}{9e^{9x} - 2e^{2x}} \\ &= \frac{3}{9 - 2} \\ &= \frac{3}{7}\end{aligned}$$

(c) $\lim_{x \rightarrow 0^+} \frac{\ln(\sin(2x))}{\ln(\sin(3x))}$

Solution.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\ln(\sin(2x))}{\ln(\sin(3x))} &= \frac{-\infty}{-\infty} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0^+} \frac{\frac{2 \cos(2x)}{\sin(2x)}}{\frac{3 \cos(3x)}{\sin(3x)}} \\ &= \lim_{x \rightarrow 0^+} \frac{2 \cos(2x) \sin(3x)}{3 \cos(3x) \sin(2x)} \\ &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0^+} \frac{-2 \sin(2x) \cdot 2 \cdot \sin(3x) + 2 \cos(3x) \cos(3x) \cdot 3}{-3 \sin(3x) \cdot 3 \cdot \sin(3x) + 3 \cos(3x) \cos(3x) \cdot 2} \\ &= \frac{0 + 2 \cdot 3}{0 + 3 \cdot 2} \\ &= 1\end{aligned}$$

(d) $\lim_{x \rightarrow 0} \frac{\ln(\cos(2x))}{\ln(\cos(3x))}$

Solution.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\ln(\cos(2x))}{\ln(\cos(3x))} &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0} \frac{\frac{2 \sin(2x)}{\cos(2x)}}{\frac{3 \sin(3x)}{\cos(3x)}} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin(2x) \cos(3x)}{3 \sin(3x) \cos(2x)} \\ &= \frac{0}{0} \\ \text{(L'hospital's)} &= \lim_{x \rightarrow 0} \frac{2 \cos(2x) \cdot 2 \cdot \cos(3x) + 2 \sin(2x) (-\sin(3x) \cdot 3)}{3 \cos(3x) \cdot 3 \cdot \cos(2x) + 3 \sin(3x) (-\sin(2x) \cdot 2)} \\ &= \frac{2 \cdot 2 + 0}{3 \cdot 3 + 0} \\ &= \frac{4}{9}\end{aligned}$$

(e) $\lim_{x \rightarrow 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1} \right)$

Solution.

See Page 309 Example 7.

Optimization

2. Find the dimensions of a rectangle with area 512 m^2 whose perimeter is as small as possible.

Solution.

$$P = 2x + 2y$$

$$A = x \cdot y = 512 \implies y = \frac{512}{x}$$

$$\begin{aligned} P &= 2x + 2y \\ &= 2x + 2\left(\frac{512}{x}\right) \\ &= 2x + \frac{1024}{x} \end{aligned}$$

$$P'(x) = 2 - \frac{1024}{x^2} = 0 \implies x^2 - 1024 = 0 \implies x = 32$$

Check if $x = 32$ is a min:

$$P''(x) = \frac{2048}{x^3} \implies P''(32) > 0 \implies x = 32 \text{ is a min}$$

Hence, the dimensions of a rectangle whose perimeter is as small as possible is

$$32 \times \frac{512}{32} = 32 \times 16$$

3. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Solution.

$$\text{Length of fencing} = L = 5x + 2y = 750$$

$$A = x \cdot y$$

$$5x + 2y = 750 \implies y = \frac{750 - 5x}{2}$$

$$A = xy = x \left(\frac{750 - 5x}{2} \right) = 375x - \frac{5x^2}{2}$$

$$A'(x) = 375 - 5x = 0 \implies x = 75$$

Check if $x = 75$ is a max:

$$A''(x) = -5 < 0 \text{ for all } x \implies x = 75 \text{ is a max}$$

Hence, the largest possible total area is

$$A = 375(75) - \frac{5(75)^2}{2} = 14062.5 \text{ ft}^2$$

4. Find the point on the line $y = 5x + 2$ that is closest to the origin.

$$y = 5x + 2$$

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

$$\begin{aligned} f(x) &= d^2 = x^2 + y^2 \\ &= x^2 + (5x+2)^2 \\ &= 26x^2 + 20x + 4 \end{aligned}$$

$$f'(x) = 52x + 20 = 0 \implies x = -\frac{5}{13}$$

Check if $x = -\frac{5}{13}$ is a min:

$$f''(x) = 52 > 0 \text{ for all } x \implies x = -\frac{5}{13} \text{ is a min}$$

Hence, the point that minimizes the distance between $y = 5x + 2$ and the origin $(0, 0)$ is

$$\left(-\frac{5}{13}, 5\left(-\frac{5}{13} + 2\right)\right) = \left(-\frac{5}{13}, \frac{1}{13}\right)$$

Antiderivatives and Definite Integrals

5. Find the most general antiderivatives of the following functions (use C as any constant).

(a) $f(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$

Solution.

$$F(x) = \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{5}x^4 + C$$

(b) $f(t) = \frac{t^4 + 3\sqrt{t}}{t^2}$

Solution.

$$f(t) = t^2 + 3t^{-3/2} \implies F(t) = \frac{t^3}{3} + \frac{3t^{-1/2}}{-1/2} = \frac{t^3}{3} - \frac{6}{\sqrt{t}} + C$$

(c) $g(\theta) = \cos \theta - 5 \sin \theta$

Solution.

$$G(\theta) = \sin(\theta) + 5 \cos(\theta) + C$$

6. Find a function $f(x)$ satisfying the following conditions:

(a) $f'''(x) = \cos(x)$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$

Solution.

$$f''(x) = \sin(x) + C$$

$$f''(0) = \sin(0) + C = 3 \implies C = 3$$

$$f''(x) = \sin(x) + 3$$

$$f'(x) = -\cos(x) + 3x + D$$

$$f'(0) = -\cos(0) + 0 + D = 2 \implies -1 + D = 2 \implies D = 3$$

$$f'(x) = -\cos(x) + 3x + 3$$

$$f(x) = -\sin(x) + \frac{3x^2}{2} + 3x + E$$

$$f(0) = 0 + 0 + 0 + E = 1 \implies E = 1$$

$$f(x) = -\sin(x) + \frac{3x^2}{2} + 3x + 1$$

(b) $f''(x) = 2 - 12x$, $f(0) = 9$, $f(2) = 7$

$$f'(x) = 2x - 6x^2 + C$$

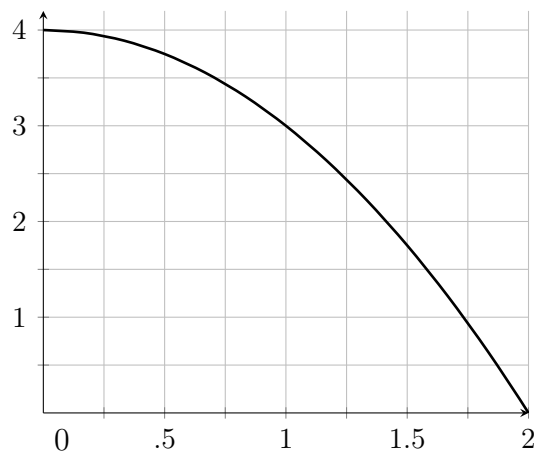
$$f(x) = x^2 - 2x^3 + Cx + D$$

$$f(0) = 0 - 0 + 0 + D = 9 \implies D = 9$$

$$f(2) = 4 - 2(8) + C(2) + 9 = 7 \implies C = 5$$

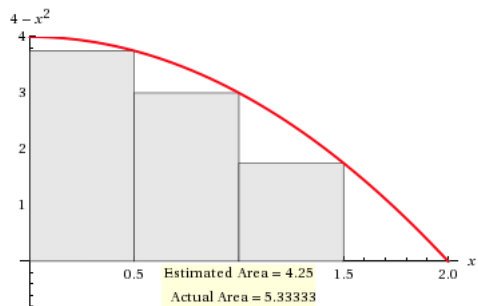
$$f(x) = x^2 - 2x^3 + 5x + 9$$

7. The graph of $y = 4 - x^2$ over the interval $[0, 2]$ is given below.



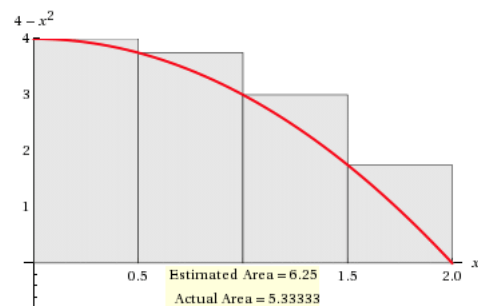
(a) Estimate the area under the graph over $[0, 2]$ using 4 rectangles and right-endpoints. Sketch the rectangles in the graph above.

Solution.



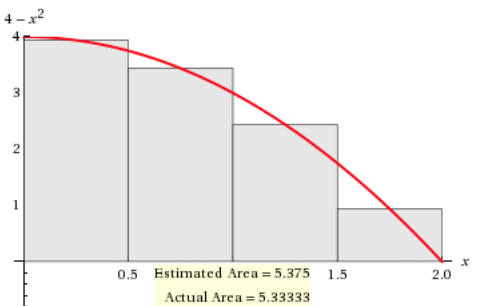
(b) Estimate the area under the graph over $[0, 2]$ using 4 rectangles and left-endpoints.

Solution.



(c) Estimate the area under the graph over $[0, 2]$ using 4 rectangles and midpoints.

Solution.



(d) Calculate the exact area under $y = 4 - x^2$ over $[0, 2]$ by going to Wolframalpha.com and typing in “integral from 0..2 of $4 - x^2$ ”. Which of the approximations in (a), (b), and (c) is closest to this?

Solution.

See the above graphs for the exact value.

Extra Credit: If an object is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m}),$$

where g is the acceleration due to gravity and c is a positive constant describing air resistance.

(a) Calculate $\lim_{t \rightarrow \infty} v$. (*You do not need L'Hospital's Rule to evaluate this limit.*) What is the meaning of this limit?

(b) For fixed t , use L'Hospital's Rule to calculate $\lim_{c \rightarrow 0^+} v$.

Solution.

(a)

$$\lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{mg}{c} (1 - 0) = \frac{mg}{c}$$

This means that as time progresses off to infinity, the velocity of the object approaches the constant value of mg/c . This is an interesting calculation, since one would think that the velocity of a falling object would keep increasing without bound. However, this calculation shows that the object eventually reaches a maximum velocity and then remains at that constant velocity. This constant velocity that the object approaches is called its **terminal velocity**.

(b)

$$\lim_{c \rightarrow 0^+} v = \lim_{c \rightarrow 0^+} \frac{mg - mge^{-ct/m}}{c} = \frac{0}{0}.$$

Using L'Hospital's Rule [with respect to c],

$$\begin{aligned} \lim_{c \rightarrow 0^+} \frac{mg - mge^{-ct/m}}{c} &= \lim_{c \rightarrow 0^+} \frac{(t/m)mge^{-ct/m}}{1} \\ &= \lim_{c \rightarrow 0^+} tge^{-ct/m} \\ &= gt \text{ m/s} \end{aligned}$$